Developing Mathematical Thinking

Why do kids have such a difficult time understanding math? Why do students struggle with place value, division, and fractions? What is the best approach for helping children learn mathematics with meaning? How important is memorizing basic facts? These are just some of the questions parents and teachers ask. This newsletter has been created, in part, to address these types of questions. Math Success is dedicated to providing insight for parents and teachers into how children learn mathematics and become mathematical thinkers. In each issue, we will focus on one mathematical topic, demonstrate student strategies, present a problem to work on, highlight a children’s book, and more.

The Center for Developing Mathematical Thinking or CDMT is an organization dedicated to assisting teachers, parents, and children in learning how to think mathematically. The Developing Mathematical Thinking or DMT is an approach based on current research about how children develop a deep and well-connected understanding of mathematics. We study how students develop mathematical ideas over time and help parents and teachers build learning environments to best promote this development. In the section “Supporting an Understanding of Addition” you will see how this approach works for one topic. In future newsletters, we will highlight other mathematical topics.

Supporting an Understanding of Addition

Word problems offer students much more opportunities to understand addition. They allow students to make multiple connections, build meaning for the operation of addition and develop problem solving skills. Here is a task that can be given to Kindergarteners or can be modified easily for older students.

**TASK:** Alex has 7 fish in his fish tank. He gets 4 more fish for his birthday. How many fish does he have now?

Initially, young children will solve this problem by **directly modeling,** in order, each set in the problem. They tend to use manipulatives (e.g., cubes or counters to represent the fish) and will count 1, 2, 3, 4, 5, 6, 7 fish. Next, they will count 1, 2, 3, 4 more fish. And finally, they will join or push all the cubes together and recount all: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11. Eleven fish!

As students mature they will eventually be able to retain one of the numbers. This type of strategy is called **counting** on or counting back. In this case a
student will say, “7 fish” and then count the next four items using cubes or their fingers as placeholders. They will say, “8” (holding up 1 finger), “9, 10, 11” (holding up the 4th finger).

Once students are able to hold a number in their head, they are ready to begin understanding the idea of place value (e.g., 14 is a 10 and 4). Students develop these ideas through using and experimenting with numbers and manipulatives while solving many different types of problems.

For young children, the most advanced strategy involves using derived facts. This means that a student knows a fact and can use that fact to solve the problem. In this case a student might know that 7 and 3 make 10 and one more is 11. Here the child broke the 4 into a 3 and 1 and used the 3 along with the 7 to make 10. This strategy is called decomposing (or breaking down) numbers. Its use indicates that a child is both flexible in how she approaches new problems and that she is learning and using basic facts. Most often, students by the end of first grade have moved into this stage. However, this does not mean that the other strategies will not be used to solve challenging problems. Even mathematicians when solving new and difficult problems will revert back to simplified versions of the problem and even draw pictures or diagrams to get a sense of the process.

Notational Systems

Encouraging children to decompose or break numbers apart is a cornerstone in building number sense and making meaning of place value.

Children who know how to decompose numbers develop flexible, mathematically powerful ways of solving problems. Two notational systems that illustrate the ways in which children decompose numbers are tree diagrams and arrow language.

Initially, children are asked to decompose numbers into two addends. When using tree diagrams, students write the number and then draw two branches below it, writing the two addends below. Tree diagrams are simple visuals that children use to demonstrate relationships between numbers. They help children build number sense and an initial foundation of place value. Below are examples of decomposing the numbers seven and twelve using tree diagrams.

![Tree Diagrams](image)

Children can also use decomposition and tree diagrams to help them flexibly solve more advanced mathematical problems. For instance, when you solve 12 + 12, you might decompose the 12s and add the 10s first to get 20 and then add the 2’s to get 4. Adding these two numbers gives you the total of 24. Or, you might decompose the 12 into an 8 and 4. This would allow you to add 12 and 8 and get 20 and then 4 more to get 24.

![Arrow Language](image)

Arrow language can help students keep track of how they decompose numbers in order to find solutions. Given the same problem as above, students’ strategies might look like the following:
Imagine if you were a frog, how far could you jump?

This book is about ratio and proportion. The book begins by posing the statement, "If you hopped like a frog...". The idea is to think about how far you could jump if you were frog. Imagine if the frog was 1 inch long and could jump 7 inches. The ratio of the length of the frog compared to the length it could jump is 1:7. So, now imagine, if you are 5 feet tall (or a length of 5 feet) and you were a frog, how far might you jump? Holding the ratio constant, you would be able to jump 7 times your length (5 x 7) or 35 feet.

Ask your child how far he or she would be able to jump. If your child is 4 feet they would jump (4 x 7) or 28 feet.

In the book, Schwartz finishes the statement by saying, "you could jump from home plate to first base in one mighty leap." The mathematics used to determine this is a ratio such as 3 inches to 6 feet or a frog that is 3 inches long and can jump 6 feet. He assumes the child reading the book is around 4 feet tall, which means they could jump around 90 feet – the distance between home plate and first base on a major league field.

One tool to help understand and solve ratio problems is the ratio table. Start by labeling the ratio table and, then, placing the given ratio into the first column. The table is multiplicative, which means you can multiply any ratio in the table by the same number. Continue this process until you find the ratio with the number you are looking for.

The book is full of other wonderful proportions – buy it or check it out from the library. Children of all ages love to imagine the leaps. And while you are with your children, you can make up others yourself. Here are three ideas to get you started:

1. If you swam as fast as a trout, how fast could you swim?
2. Do you think you walk faster than a turtle? Well, if you were a turtle, how fast could you walk?
3. If you were a big brown bat, how fast could you fly?

For each of these tasks, look up the necessary information on the Web or find it in an encyclopedia. How fast do fish swim, how fast do brown bats fly? If you don’t want to solve these problems, just have fun making up your own ratio problems and simply imagine what it might be like. Running as fast as an elephant or flying as fast as a falcon might surprise you. Enjoy the book and the mathematics and most importantly, “Have fun!”

Possible solutions on page 6
Problem to Solve

In each of our newsletters we will highlight student solutions to fun and interesting math problems. In the future, we will include the problems along with solutions that students have submitted to us. In this, our inaugural edition, we have included two example problems with sample student solutions, just to illustrate what we mean.

Grades K–2
Write down everything you can about the number 6. What about the number 12?

Example Student Responses

- Six can be made of three and three.
- Six can be made of two, two and two.
- Six is a five and another 1.
- Six is bigger than 5.
- I have to use two hands to show six on my fingers.
- Twelve is made up of a ten and 2 ones.
- 12 can be broken into 5 and 7.
- Twelve can be made up of three groups of four (or four groups of three).
- It is how eggs are packaged.
- It is a dozen.
- Half of it is six.
- It is how many inches are in one foot.

“\textit{A problem is a task that you are working on where the path and solution is not obvious.}”

Grades 3–5
What do you know and what can you find out about the multiples of 3: 3, 6, 9, 12, 15, 18, 21, ...?

Example Student Responses

- They are all divisible by three
- The sums of the digits are 3
- Some are odd and some are even
- If you add two of them, you get another
- They can all be expressed as 3 times another number

Now you try
These are for you to work on during the upcoming weeks. Have fun! And be sure to submit your ideas to: jlbrende@aol.com so that we can highlight them in our next newsletter!

Grades K–2
What do you know and what can you find out about the number 18? What about the number 80? What about 180?

Grades 3–5
If I count by two's, I will land on 100 and 1,000. If I count by three's, I won't land on either. What can you count by so that you don't land on 100 but you do land on 1,000? Be sure to explain your method for figuring out the answer.
Math Success

Research Highlight

Learning Mathematics with Understanding

Yesterday: Focus on Memorization
Most of us probably remember memorizing our times tables, practicing long division for hours at our dining room tables, or completing math worksheets that included very few words. Some of us have fond memories of these experiences. For others, however, memories of learning mathematics conjure up feelings of anxiety and disappointment. Whether the memories are positive or negative, most of us were expected to memorize facts and rules for getting the right answers to problems. We were expected to complete those problems quickly, with little thought given to why the rules worked.

Today: Focus on Understanding
Research shows that students who memorize facts or procedures without understanding them often are not sure when and how to use what they know. Being “fast” at math skills is not very helpful to children or adults who do not know when or how to use their skills in the real world. This kind of mathematical learning can be thought of as a “house of cards” – with many gaps and holes, precariously balanced and unstable. In addition to causing difficulty in real world settings, these holes and gaps will likely make learning mathematics more difficult and even impossible in later grades.

Today, however, children are learning mathematics in new and exciting ways. For many reasons, your child’s mathematics education may be very different from yours. One reason is that children (and adults) learn best when they have to describe and justify their ideas. To assist with this, your child may be encouraged to solve problems in different ways. A second reason is that when children solve problems using strategies that make sense to them, they are more able to build more sophisticated ideas to solve new and more difficult problems. Eventually, a child’s own solution strategy can be used to develop understanding of more formal mathematical rules and facts.

What is Being Done?
The DMT project helps students and teachers learn mathematical skills and procedures with understanding. Research indicates that children who develop an understanding of mathematical ideas are more successful in learning mathematics. They are also more likely to remember what they have learned and be able to apply what they have learned in new situations.

Keeping a careful eye on the horizon – developing a deep understanding of mathematical ideas – teachers skillfully guide their students through various stages of understanding and development. Because children develop at different rates and must progress through various stages, teachers must be able to guide their students through various stages of mathematical development as well. In this newsletter we have highlighted various

“Research shows that students who memorize facts or procedures without understanding them often are not sure when and how to use what they know.”

Research Highlight continued on page 6
stages of understanding for addition. In future newsletters we will highlight various stages for other mathematical ideas such as subtraction, multiplication and division.

With careful guidance and support from teachers, children can develop a deep understanding of mathematical ideas and master the skills necessary to meet the demands of tomorrow’s workforce.

References

Possible solutions from page 3

![Possible solutions image]

Resources

Websites
- www.nctm.org
- csi.boisestate.edu/dmt.htm
- www.mathematicallysane.com

Articles

Children’s Books